Christ Church
Grammar School

2017
Investigation 1

Year 12 METHODS
Part One:
Take home investigation

Student name ___________________________________

Teacher name ___________________________________

Time allowed for this section
Working time for this section: 2 weeks

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, and up to three calculators approved for use in the WACE examinations

Important note to candidates
This investigation has 2 parts: a take home investigation that the student needs to research and complete at home, and an in-class validation that will be completed in test conditions during class.

The take home investigation will not be assessed; however, answers will be provided. The take home investigation needs to be completed prior to the validation.

The in-class validation will be assessed and it constitutes **6%** of the course mark.

This paper is the take home investigation.
It is handed out on Thursday 2 Feb 2017.

The in-class validation will be on Thursday 16 Feb 2017.
The in-class validation will comprise a calculator-free section and a calculator-assumed section.

Instructions to candidates

1. Write your answers in this Question/Answer Booklet.

2. Answer all questions.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that **you do not use pencil**, except in diagrams.

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Aim of Investigation

The aim of this investigation is determine the derivatives of $\sin x$ and $\cos x$ from first principles and apply the results established in routine differentiation problems.

Learning Objectives

At the end of this investigation, you should be able to:

- Establish the limit $\frac{\sin h}{h} \to 1$ as $h \to 0$ using inequalities, graphically and numerically
- Establish the limit $\frac{1-\cos h}{h} \to 0$ as $h \to 0$
- Establish from first principles that $\frac{d}{dx}(\sin x) = \cos x$
- Establish from first principles that $\frac{d}{dx}(\cos x) = -\sin x$
- Apply the derivatives of $\sin x$ and $\cos x$ in differentiation involving chain rule, product rule and quotient rule

Required Material

1. All the material contained in this booklet.
2. All the material found in:
   A.J. Sadler, Mathematics Methods Unit 3, Pages 137 to 150

Instructions to Candidates

1. To achieve the objectives of this investigation and to prepare for the validation, students will have to work through all the notes and questions specified as Required Material.
2. Additional resources can be found in:
   O.T. Lee, WACE Revision Series, Mathematics Methods Year 12, Pages 45 – 50

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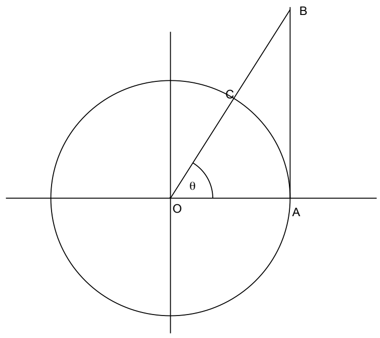
Question 1

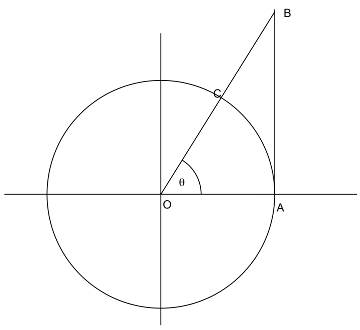
This diagram shows a unit circle. AB is a tangent to the circle and θ is measured in *radians*.



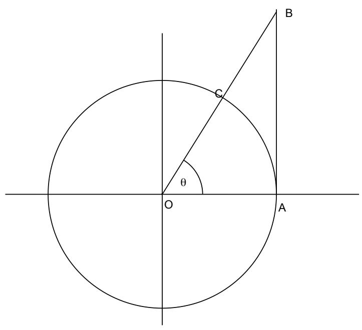
(a) Write a trigonometric expression for $|AB|$ in terms of θ, showing your reasoning.

(b) On separate diagrams of the unit circle below shade in each of the indicated areas.

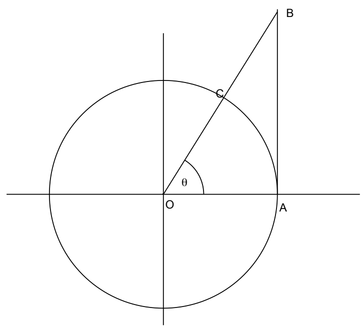
(i) Area of triangle OAC



(ii) Area of sector OAC



(iii) Area of triangle OAB



(c) Write expression for each of the following areas in terms of θ.

(i) Area of triangle OAC

(ii) Area of sector OAC

(iii) Area of triangle OAB

(d) Complete the table below calculating the areas of each shape to six decimal places:

θ in radians	Area of triangle OAC	Area of sector OAC	Area of triangle OAB
$\frac{\pi}{16}$			
$\frac{\pi}{32}$			
$\frac{\pi}{64}$			
$\frac{\pi}{128}$			
$\frac{\pi}{256}$			

(e) Place in ascending order the expressions for the area of each shape (i.e. from smallest area to largest area)

(f) Use your results to show that:

$\sin\theta < \theta < \tan\theta$ for $0 < \theta < \frac{\pi}{2}$

(g) Use the result from (f) to establish the following inequalities:

(i) $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$

(ii) $\cos\theta < \dfrac{\sin\theta}{\theta} < 1$

(h) Use the table of values in (d) to compare the areas of each figure as $\theta \to 0$, and comment on their results.

(i) Consider the value of $\cos\theta$ as $\theta \to 0$. Using the results established so far, deduce the limit that $\dfrac{\sin\theta}{\theta}$ approaches as $\theta \to 0$.

(j) Validate your results by completing the table of values below.

θ	$\dfrac{\pi}{32}$	$\dfrac{\pi}{64}$	$\dfrac{\pi}{128}$	$\dfrac{\pi}{256}$
$\cos\theta$				
$\dfrac{\sin\theta}{\theta}$				

(k) Write your conclusion for the following limit:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} =$$

Question 2

(a) Show that

$$\frac{1-\cos\theta}{\theta}=\frac{\sin^2\theta}{\theta(1+\cos\theta)}$$

(b) It is given that $\lim\limits_{\theta\to 0} f(\theta)g(\theta)=\lim\limits_{\theta\to 0} f(\theta)\times\lim\limits_{\theta\to 0} g(\theta)$.

Use this information and the result from (a), show that

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\times\ \lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}$$

(c) Hence, using the results from (b) and from Question 1, show that

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$$

Question 3

The derivative of a function $f(x)$ is found from first principles as follow:

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

(a) Write down the expansion for $\sin(A+B)$.

(b) Hence write down the expansion for $\sin(x+h)$.

(c) Using the result from (b), simplify the following expression

$$\frac{\sin(x+h)-\sin x}{h}$$

(d) Using the result from (c) and the limits established in Questions 1 and 2, determine $\frac{d}{dx}\sin x$ from first principles.

The first few steps have been completed for you.

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\cos x\sin h}{h} - \frac{\sin x - \sin x\cos h}{h}\right)$$

$$=$$

Question 4

Using a similar approach to Question 3, determine $\frac{d}{dx}\cos x$ from first principles.

The first few steps have been completed for you.

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\cos x\cos h-\cos x}{h}-\frac{\sin x\sin h}{h}\right)$$

$$=$$

Question 5

The derivatives of $\sin x$ and $\cos x$ established in Questions 3 and 4 are used in the differentiation of terms involving chain rule, product rule, quotient rule.

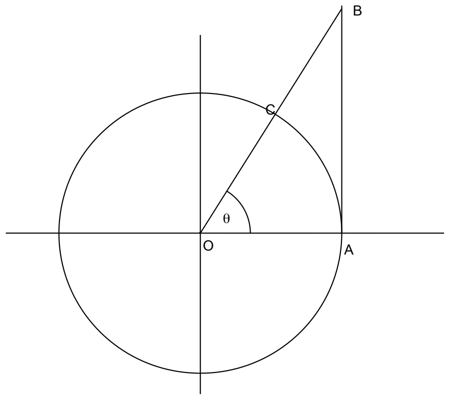
Practise this application by attempting all the questions in A.J. Sadler, Mathematics Methods Unit 3, Pages 148 – 150, Exercise 7A

End of questions

Solutions

Question 1

This diagram shows a unit circle. AB is a tangent to the circle and θ is measured in *radians*.



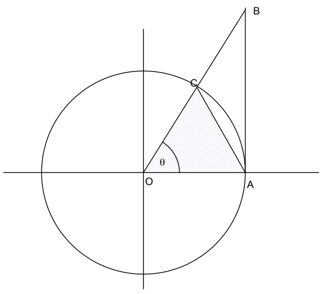
(a) Write a trigonometric expression for $|AB|$ in terms of θ, showing your reasoning.

$\tan\theta = \frac{AB}{1}$, since $AO = 1$

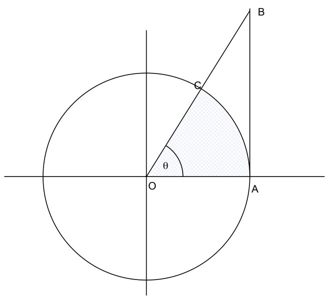
Thus, $AB = \tan\theta$

(b) On separate diagrams of the unit circle below shade in each of the indicated areas.

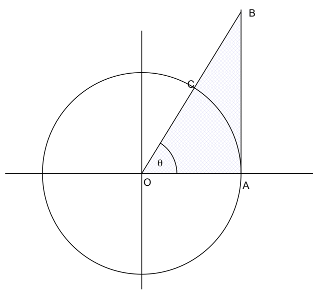
(i) Area of triangle OAC



(ii) Area of sector OAC



ii) Area of triangle OAB



(c) Write expression for each of the following areas in terms of θ.

(i) Area of triangle OAC

$$\text{Area} = \frac{1}{2}\times 1\times 1\times \sin\theta$$
$$= \frac{1}{2}\sin\theta$$

(ii) Area of sector OAC

$$\text{Area} = \frac{1}{2}\times 1\times 1\times \theta$$
$$= \frac{1}{2}\theta$$

(iii) Area of triangle OAB

$$\text{Area} = \frac{1}{2}\times 1\times 1\times \tan\theta$$
$$= \frac{1}{2}\tan\theta$$

(d) Complete the table below calculating the areas of each shape to six decimal places:

θ in radians	Area of triangle OAC	Area of sector OAC	Area of triangle OAB
$\frac{\pi}{16}$	$\text{Area} = \frac{1}{2} \times 1^2 \times \sin\frac{\pi}{16}$ $= 0.097545$	$\text{Area} = \frac{1}{2} \times 1^2 \times \frac{\pi}{16}$ $= 0.098175$	$\text{Area} = \frac{1}{2} \times 1 \times \tan\frac{\pi}{16}$ $= 0.099456$
$\frac{\pi}{32}$	0.049009	0.049087	0.049246
$\frac{\pi}{64}$	0.024534	0.024544	0.024563
$\frac{\pi}{128}$	0.012271	0.012272	0.012274
$\frac{\pi}{256}$	0.006136	0.006136	0.006136

(e) Place in ascending order the expressions for the area of each shape (i.e. from smallest area to largest area)

Area of triangle OAC < Area of sector OAC < Area of triangle OAB

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

(f) Use your results to show that:

$\sin\theta < \theta < \tan\theta$ for $0 < \theta < \frac{\pi}{2}$

Multiply the inequality in (e) throughout by 2,

$$\sin\theta < \theta < \tan\theta$$

where $\angle AOC = \theta$ and $0 < \theta < \frac{\pi}{2}$

(g) Use the result from (f) to establish the following inequalities:

(i) $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$

$$\sin\theta < \theta < \tan\theta$$

dividing throughout by $\sin\theta$ $\Rightarrow$ $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$

(ii) $\cos\theta < \dfrac{\sin\theta}{\theta} < 1$

All terms in the above inequality are positive since $0 < \theta < \frac{\pi}{2}$.

$1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$ $\Rightarrow$ $1 < \dfrac{\theta}{\sin\theta}$ and $\dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$

$1 < \dfrac{\theta}{\sin\theta}$ $\Rightarrow$ $\dfrac{\sin\theta}{\theta} < 1$ and $\dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$ $\Rightarrow$ $\cos\theta < \dfrac{\sin\theta}{\theta}$

Hence, $\cos\theta < \dfrac{\sin\theta}{\theta} < 1$

(h) Use the table of values to compare the areas of each figure as $\theta \to 0$, and comment on their results.

As $\theta \to 0$,

Area of triangle OAC < Area of sector OAC < Area of triangle OAB

And, Area of triangle OAC → Area of sector OAC → Area of triangle OAB

(i) Consider the value of $\cos\theta$ as $\theta \to 0$. Using the results established so far, deduce the limit that $\frac{\sin\theta}{\theta}$ approaches as $\theta \to 0$.

From (g)(ii), $\cos\theta < \frac{\sin\theta}{\theta} < 1$

As $\theta \to 0$, $\cos\theta \to 1$

$\cos\theta < \frac{\sin\theta}{\theta} < 1$ becomes $1 < \frac{\sin\theta}{\theta} < 1$

$\Rightarrow \frac{\sin\theta}{\theta} \to 1$

(j) Validate your results by completing the table of values below.

θ	$\frac{\pi}{32}$	$\frac{\pi}{64}$	$\frac{\pi}{128}$	$\frac{\pi}{256}$
$\cos\theta$	0.995185	0.998795	0.999699	0.999925
$\frac{\sin\theta}{\theta}$	0.998394	0.999598	0.999900	0.999925

(k) Write your conclusion for the following limit:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Question 2

(a) Show that

$$\frac{1-\cos\theta}{\theta} = \frac{\sin^2\theta}{\theta(1+\cos\theta)}$$

$$\frac{1-\cos\theta}{\theta} = \frac{1-\cos\theta}{\theta} \times \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$$

$$= \frac{\sin^2\theta}{\theta(1+\cos\theta)}$$

(b) It is given that $\lim\limits_{\theta\to 0} f(\theta)g(\theta) = \lim\limits_{\theta\to 0} f(\theta) \times \lim\limits_{\theta\to 0} g(\theta)$.

Use this information and the result from (a), show that

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \times \lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}$$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)} \qquad \text{from (a)}$$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \times \frac{\sin\theta}{(1+\cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \times \lim_{\theta\to 0}\frac{\sin\theta}{(1+\cos\theta)}$$

(c) Hence, using the results from (b) and Question 1, show that

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$$

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \times \lim_{\theta \to 0} \frac{\sin\theta}{(1 + \cos\theta)} \qquad \text{from (b)}$$

$$= 1 \times \frac{\sin 0}{(1 + \cos 0)} \qquad \text{from Question 1}$$

$$= 1 \times 0 \qquad \text{since } \sin 0 = 0$$

$$= 0$$

Question 3

The derivative of a function $f(x)$ is found from first principles as follow:

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

(a) Write down the expansion for $\sin(A+B)$.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

(b) Hence write down the expansion for $\sin(x+h)$.

$$\sin(x+h) = \sin x\cos h + \cos x\sin h$$

(c) Using the results from (b), simplify the following expression

$$\frac{\sin(x+h)-\sin x}{h}$$

$$\frac{\sin(x+h)-\sin x}{h} = \frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

(d) Using the result from (c) and the limits established in Questions 1 and 2, determine $\frac{d}{dx}\sin x$ from first principles.

The first few steps have been completed for you.

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\cos x\sin h}{h} - \frac{\sin x - \sin x\cos h}{h}\right)$$

$$= \lim_{h\to 0}\frac{\cos x\sin h}{h} - \lim_{h\to 0}\frac{\sin x - \sin x\cos h}{h}$$

$$= \cos x\lim_{h\to 0}\frac{\sin h}{h} - \sin x\lim_{h\to 0}\frac{1-\cos h}{h} \qquad \text{since } \lim_{h\to 0}\frac{\sin h}{h} = 1,\ \lim_{h\to 0}\frac{1-\cos h}{h} = 0$$

$$= \cos x\,(1) - \sin x\,(0)$$

$$= \cos x$$

Question 4

Using a similar approach to Question 3, determine $\frac{d}{dx}\cos x$ from first principles.

The first few steps have been completed for you.

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\cos x\cos h - \cos x}{h} - \frac{\sin x\sin h}{h}\right)$$

$$= \lim_{h\to 0}\frac{\cos x\cos h - \cos x}{h} - \lim_{h\to 0}\frac{\sin x\sin h}{h}$$

$$= -\cos x \lim_{h\to 0}\frac{1-\cos h}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h} \qquad \text{since } \lim_{h\to 0}\frac{\sin h}{h} = 1 \text{ , } \lim_{h\to 0}\frac{1-\cos h}{h} = 0$$

$$= -\cos x\ (0) - \sin x\ (1)$$

$$= -\sin x$$

Question 5

The derivatives of $\sin x$ and $\cos x$ established in Questions 3 and 4 are used in the differentiation of terms involving chain rule, product rule, quotient rule.

Practise this application by attempting all the questions in A.J. Sadler, Mathematics Methods Unit 3, Pages 148 – 150, Exercise 7A

End of questions